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Wrapped fivebranes and $\mathcal{N} = 2$ super Yang–Mills theory

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Abstract

We construct $D = 10$ supergravity solutions corresponding to type IIB fivebranes wrapping a two-sphere in a Calabi–Yau two-fold. These are related in the IR to the large N limit of pure $\mathcal{N} = 2$ $SU(N)$ super Yang–Mills theory. We show that the singularities in the IR correspond to the wrapped branes being distributed on a ring. We analyse the dynamics of a probe fivebrane and show that it incorporates the full perturbative structure of the gauge theory. For a class of solutions the two-dimensional moduli space is non-singular and we match the result for the corresponding slice of the Coulomb branch of the gauge theory.

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1 Introduction

It is interesting to generalise the AdS/CFT correspondence [1] to systems with less than maximal supersymmetry and hence richer dynamics. One strategy is to construct gravity duals corresponding to branes wrapping supersymmetric cycles as in [2]–[12]. This approach has been used to construct supergravity solutions corresponding in the IR to pure $\mathcal{N} = 1$ super Yang–Mills theory in $D = 4$ [3]. Here we investigate similar constructions related to pure $\mathcal{N} = 2$ super Yang–Mills theory in $D = 4$.

As in [3], the super Yang–Mills theory arises from the IIB little string theory which describes a collection of NS fivebranes in the limit of vanishing string coupling [13]. Although this theory is still somewhat mysterious, it is known that it reduces at low energies to $D = 6$ Yang–Mills theory with 16 supercharges. Dimensionally reducing on a two-cycle Σ should then give a four-dimensional field theory. In order to preserve supersymmetry, the little string theory on $\mathbb{R}^{1,3} \times \Sigma$ must be coupled to external R -symmetry currents, that is, it must be “twisted” [14]. This requires appropriately identifying the $U(1)$ spin connection on the cycle Σ with a $U(1)$ subgroup of the $SO(4)$ R -symmetry group. From a geometric standpoint, the twisting is the same as that arising when a fivebrane wraps a two-cycle Σ in a Calabi–Yau manifold. In particular, the $SO(4)$ R -symmetry corresponds to the normal bundle to the fivebrane. The Calabi–Yau condition then requires one to identify a $U(1)$ factor in the normal bundle with the structure group of the tangent bundle to Σ . In order to preserve eight supercharges, the two-cycle should be in a Calabi–Yau two-fold. By contrast the twisting considered in [3] is that associated with fivebranes wrapping two-cycles in a Calabi–Yau three-fold (with non-generic normal bundle) and hence preserves four supercharges. As discussed below, if Σ is chosen to be a two-sphere, there are no additional matter multiplets, so that at energies smaller than the scale set by the radius of the sphere we have pure $\mathcal{N} = 2$, $D = 4$ super Yang–Mills theory.

To construct suitable gravity duals we follow the strategy set out in [2]. We first construct solutions in a truncated $D = 7$ gauged supergravity theory and then uplift to obtain solutions in $D = 10$. The solutions are determined by two parameters, one of which is the expectation value of the dilaton. For a class of solutions we show that the singularities that appear in the IR correspond to the wrapped branes being distributed or “smeared” over a ring, indicating which part of the Coulomb branch of $\mathcal{N} = 2$ super Yang–Mills theory the solutions are related to. We show that the supergravity solution possesses an $SU(2)$ symmetry corresponding to the $SU(2)$ R -symmetry of the gauge theory. The supergravity solution also possesses a $U(1)$

isometry corresponding to the $U(1)$ R -symmetry and we argue that this is broken to the anomaly free Z_{4N} subgroup by string-worldsheet instantons as in [3].

To obtain further insight into the solutions we analyse the dynamics of a probe fivebrane. We find that the dynamics is governed by a holomorphic prepotential on a two-dimensional moduli space that incorporates the exact perturbative effects of the gauge theory. In addition, for a class of singular solutions, we remarkably find a regular moduli space.

Other approaches to studying the large N limit of $\mathcal{N} = 2$ super Yang–Mills theory have appeared in [15]–[27]. Our solution has some similarities with that of [17] on which we will comment at the end of the paper.

The remainder of the paper is structured as follows. In section 2, we discuss the twisting of the wrapped little string theory and how it leads to $\mathcal{N} = 2$ super Yang–Mills theory in the IR. Section 3 presents the supergravity solutions and analyses their properties. Section 4 analyses the dynamics of the probe fivebranes and section 5 makes a comparison with gauge theory. Section 6 briefly concludes.

2 NS fivebranes wrapped on S^2 and $\mathcal{N} = 2$ Yang–Mills theory

Let us start by recalling the IIB little string theory arising from a collection of flat NS fivebranes. This corresponds to the limit where the string scale α' is kept fixed while the string coupling goes to zero [13]. In the IR the theory flows to six-dimensional Yang–Mills theory. For large N the theory can be studied using supergravity [28, 29] by analysing the near horizon limit of N NS-branes¹ which is given, with $\alpha' = 1$, by

$$\begin{aligned} ds^2 &= d\xi_{1,5}^2 + N (d\rho^2 + d\Omega_3^2), \\ e^{-2\Phi} &= e^{-2\Phi_0} e^{2\rho}, \end{aligned} \tag{1}$$

where $d\xi_{1,5}^2$ is the Minkowski metric on $\mathbb{R}^{1,5}$ and the integral of the three-form $H/4\pi^2$ on the three-sphere is N . With regard to supersymmetry, the two chiral $SO(9,1)$ spinors of IIB string theory each decompose under an $SO(5,1) \times SO(4)$ subgroup as $\mathbf{16}_+ \rightarrow (\mathbf{4}_+, \mathbf{2}_+) + (\mathbf{4}_-, \mathbf{2}_-)$, where the subscripts refer to chirality. The $D = 10$ Majorana condition implies that each representation is a symplectic-Majorana spinor in $D = 6$. The fivebrane preserves the $\mathbf{2}_+$ part of one spinor and the $\mathbf{2}_-$ part of the other. In other words, the spin content of the preserved supersymmetries is the same as that of the decomposition of a single $\mathbf{16}_+$.

¹At some scales the S-dual D5-brane solutions are more appropriate [28].

Now suppose the fivebrane is wrapped on a two-sphere. In the IR, at length scales much larger than the size of the sphere, we have a four-dimensional field theory. The supersymmetries then have a further decomposition under $SO(3, 1) \times SO(2) \times SO(4)$. To preserve eight supercharges we split the R -symmetry group via $SO(4) \rightarrow SO(2)_1 \times SO(2)_2$ and identify the $SO(2)$ spin connection of S^2 with one of the $SO(2)$ factors, say $SO(2)_1$. The preserved supersymmetries are singlets under the diagonal $SO(2)$ and it is straightforward to see that this twisting leaves us with eight supercharges or $\mathcal{N} = 2$ in $D = 4$. Geometrically it is the same twisting that arises in the local description of a fivebrane wrapping a sphere within a Calabi–Yau two-fold. There the R symmetry corresponds to the symmetry group of the normal bundle. This is naturally split into a $SO(2)_1$ describing normal directions to the brane within the Calabi–Yau manifold and an $SO(2)_2$ describing the remaining flat normal directions. The Calabi–Yau condition, that the two-fold has $SU(2)$ and not $U(2)$ holonomy, requires the identification of the first $SO(2)_1 = U(1)$ sub-bundle with the $SO(2)$ tangent bundle of the two-sphere, exactly as discussed above.

The four scalars in the little string theory transform as a $\mathbf{4}$ of $SO(4)$ and hence as $(\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2})$ of $SO(2)_1 \times SO(2)_2$. After twisting the former do not have any zero-modes on the two-sphere, since the two-sphere is rigid within the Calabi–Yau manifold, while the latter give rise to two massless $D = 4$ scalars. The components of the gauge field on the two-sphere have no zero-modes so upon dimensional reduction one has simply a $D = 4$ gauge field. These zero modes and their fermionic partners thus comprise the fields of pure $D = 4$, $\mathcal{N} = 2$ $U(1)$ super Yang–Mills theory. Generalising to N fivebranes we get $SU(N)$ gauge group. Note that if one were to consider fivebranes wrapped on a genus g Riemann surface, there would be zero-modes arising from deformations of the two-cycle within the Calabi–Yau manifold, as well as from the gauge field and one would find g additional hypermultiplets in the adjoint representation.

3 Supergravity solution

The supergravity solutions for the wrapped NS fivebranes only involve NS fields and hence can be constructed in $\mathcal{N} = 1$ supergravity in $D = 10$. The solutions for wrapped D-fivebranes then follow by S-duality. Following the strategy set out in [2], we first construct the solutions in a truncated theory in $D = 7$ and then uplift them to $D = 10$.

A $D = 7$, $SO(4)$ gauged supergravity is expected to arise from the consistent

truncation of $\mathcal{N} = 1$ supergravity on a three-sphere with the $SO(4)$ gauge-fields arising from the isometries of the sphere. This should be the minimal $SU(2)$ gauged supergravity [30] coupled to an $SU(2)$ gauge multiplet. A Kaluza–Klein ansatz for the bosonic fields was presented in [31] which reduces the bosonic equations of motion of $\mathcal{N} = 1$ supergravity to $D = 7$ field equations. We would like to find solutions preserving 1/4 of the supersymmetry. However, since we do not have the full reduced supersymmetric theory², we cannot check for supersymmetry directly in $D = 7$. Rather we first construct and solve a set of first-order seven-dimensional BPS equations, then use [31] to uplift to give solutions in $D = 10$ and finally directly check that the $D = 10$ solutions preserve 1/4 of the supersymmetry both in $\mathcal{N} = 1$ and type II supergravity. Consequently, logically it is possible to skip the details of the seven-dimensional derivation given in the next subsection and start simply with the explicit uplifted $D = 10$ solution given at the beginning of section 3.2.

3.1 The gravity solution in $D = 7$

The bosonic field content of the putative $D = 7$ $SO(4)$ gauged supergravity consists of a metric, $SO(4)$ gauge fields, a three-form and ten scalar fields parametrised by a symmetric four by four matrix T_{ij} . To construct supergravity solutions corresponding to the twistings discussed above we split $SO(4) \rightarrow SO(2)_1 \times SO(2)_2$ and set all gauge-fields to zero except those corresponding to $SO(2)_1$. The ansatz for the scalar fields is given by

$$T_{ij} = e^{y/4} \text{diag}(e^x, e^x, e^{-x}, e^{-x}) \quad (2)$$

and we set the three-form to zero. With these simplifications, the equations of motion given in eq. (27) in [31] can be obtained from the Lagrangian

$$\mathcal{L} = \sqrt{g} \left(R - \frac{5}{16} \partial_\mu y \partial^\mu y - \partial_\mu x \partial^\mu x - \frac{1}{4} e^{-2x-y/2} F_{\mu\nu} F^{\mu\nu} + 4g^2 e^{y/2} \right). \quad (3)$$

For the metric and gauge field we choose the ansatz

$$\begin{aligned} ds^2 &= e^{2f(r)} (d\xi^2 + dr^2) + a^2(r) d\Omega_2^2 \\ F &= \frac{1}{g} \omega_2 \end{aligned} \quad (4)$$

where $d\xi^2$ is the Minkowski metric on $\mathbb{R}^{1,3}$, $d\Omega_2^2 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2$ and $\omega_2 = \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\phi}$ are the metric element and the volume form of S^2 , respectively. Note that as in

²Note Added: After this work was completed we became aware of ref. [32] where this $D = 7$ gauged supergravity was constructed.

similar studies it is straightforward to also obtain solutions for hyperbolic space H^2 , or a quotient thereof. These solutions would be related to duals of $\mathcal{N} = 2$ gauge theories with matter. The ansatz for the gauge field can be written in terms of the connection as $A = g^{-1} \cos \tilde{\theta} d\tilde{\phi}$, which is proportional to the spin-connection on the tangent bundle to the sphere, and so incorporates the desired twisting. The gauge field equation of motion is automatically satisfied, whereas the scalar field equations and Einstein equations give

$$\begin{aligned}
\ddot{x} + \left(3\dot{f} + 2\frac{\dot{a}}{a}\right)\dot{x} &= -\frac{1}{2g^2a^4}e^{2f-2x-y/2} \\
\ddot{y} + \left(3\dot{f} + 2\frac{\dot{a}}{a}\right)\dot{y} &= -\frac{1}{5}e^{2f}\left(\frac{2}{g^2a^4}e^{-2x-y/2} + 16g^2e^{y/2}\right) \\
\ddot{f} + \left(3\dot{f} + 2\frac{\dot{a}}{a}\right)\dot{f} &= \frac{1}{10}e^{2f}\left(\frac{1}{g^2a^4}e^{-2x-y/2} + 8g^2e^{y/2}\right) \\
4\ddot{f} + 2\frac{\ddot{a}}{a} - 2\dot{f}\frac{\dot{a}}{a} &= \frac{1}{10}e^{2f}\left(\frac{1}{g^2a^4}e^{-2x-y/2} + 8g^2e^{y/2}\right) - \frac{5}{16}\dot{y}^2 - \dot{x}^2 \\
\frac{\ddot{a}}{a} + 3\frac{\dot{a}}{a}\dot{f} + \frac{\dot{a}^2}{a^2} &= e^{2f}\left(\frac{1}{a^2} - \frac{2}{5g^2a^4}e^{-2x-y/2} + \frac{4}{5}g^2e^{y/2}\right)
\end{aligned} \tag{5}$$

with dots denoting derivatives with respect to r .

When we uplift to $D = 10$, using the formulae in [31], to describe an NS fivebrane configuration we want the warp factor in the string frame multiplying the unwrapped world-volume directions $\{\xi^i\}$ to be unity. This leads us to set

$$e^{2f+y/2} = 1, \tag{6}$$

which is consistent with (5). To find solutions to the remaining equations it is convenient to introduce new variables

$$\begin{aligned}
e^{2A} &= e^{3f}a^2 \\
e^{2h} &= e^{-2f}a^2.
\end{aligned} \tag{7}$$

We then find that the equations of motion can be derived from an effective action whose Lagrangian is given by

$$\begin{aligned}
L &= e^{2A} \left[4\dot{A}^2 - 2\dot{h}^2 - \dot{x}^2 - V \right] \\
V(h, x) &= -2e^{-2h} + \frac{1}{2g^2}e^{-4h-2x} - 4g^2.
\end{aligned} \tag{8}$$

providing that in addition we demand that the Hamiltonian, given by

$$H = \frac{1}{4}e^{-2A} \left(\frac{1}{4}p_A^2 - \frac{1}{2}p_h^2 - p_x^2 \right) + e^{2A}V, \tag{9}$$

vanishes. One can then solve the system by finding a set of first-order BPS equations such that Lagrangian can be written as a sum of squared terms. That is to say, one can reduce the equations of motion to a first-order Hamiltonian system, with an associated Hamilton–Jacobi equation [33]. By choosing an ansatz for the principal function $F = e^{2A}W(h, x)$ we find that W obeys

$$V = \frac{1}{4} \left(\frac{1}{2} \partial_h W^2 + \partial_x W^2 - W^2 \right). \quad (10)$$

An analytic solution to (10) is given by

$$W = -(4g \cosh x + \frac{1}{g} e^{-2h-x}). \quad (11)$$

The remaining Hamiltonian equations give rise to the following first-order BPS equations,

$$\begin{aligned} \dot{A} &= \frac{1}{4} W = -g \cosh x - \frac{1}{4g} e^{-2h-x} \\ \dot{h} &= -\frac{1}{4} \partial_h W = -\frac{1}{2g} e^{-2h-x} \\ \dot{x} &= -\frac{1}{2} \partial_x W = 2g \sinh x - \frac{1}{2g} e^{-2h-x}. \end{aligned} \quad (12)$$

We expect that these equations imply the solution is supersymmetric. This will be implicitly verified when we explicitly show that the uplifted solutions in $D = 10$ preserve 1/4 of the supersymmetry.

We can now solve the BPS equations by using

$$z = e^{2h}, \quad (13)$$

as a new radial variable and we find³

$$\begin{aligned} e^{-2x} &= 1 - \frac{1 + k e^{-2g^2 z}}{2g^2 z}, \\ e^{2A+x} &= z e^{2g^2 z}, \end{aligned} \quad (14)$$

where k is an integration constant (the other integration constant can be removed by a coordinate transformation in the metric (4)). We will see in the next section that the parameter k labels different flows from the UV to the IR. In Fig. 1 we have plotted the various orbits of e^{-2x} corresponding to different ranges of k .

³For negatively curved two-cycles, the solution is $e^{-2x} = 1 + (1 + k e^{2g^2 z})/2g^2 z$ and $e^{2A+x} = z e^{-2g^2 z}$.

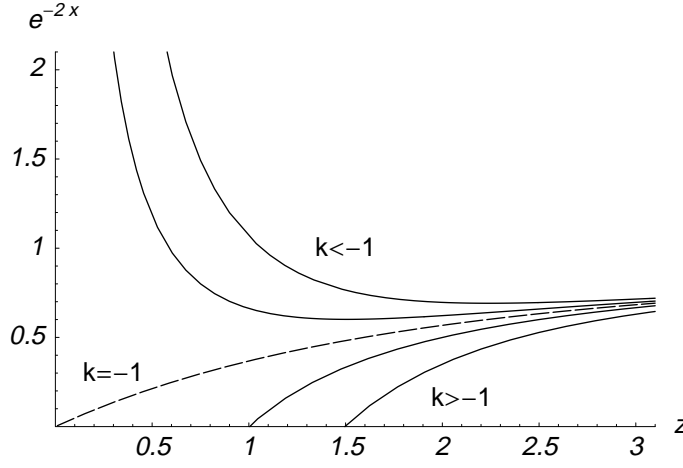


Figure 1: Behaviour of e^{-2x} for different values of k .

3.2 The supergravity solution in $D = 10$

Using the uplifting formulae given in [31] we can extract the $D = 10$ metric, dilaton and NS three-form. In the string frame our family of solutions read

$$\begin{aligned}
 ds^2 = & d\xi_{1,3}^2 + z(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2) + g^2 e^{2x} dz^2 + \frac{1}{g^2} d\theta^2 \\
 & + \frac{1}{g^2 \Omega} e^{-x} \cos^2 \theta (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi})^2 + \frac{1}{g^2 \Omega} e^x \sin^2 \theta d\phi_2^2,
 \end{aligned} \tag{15}$$

where

$$\Omega = e^x \cos^2 \theta + e^{-x} \sin^2 \theta. \tag{16}$$

The coordinates $\{\xi^i\}$, $i = 0, \dots, 3$, parameterise the unwrapped world volume directions, $\{\tilde{\theta}, \tilde{\phi}\}$ the wrapped ones, and $\{\theta, \phi^1, \phi^2\}$ are the angles of the squashed and twisted three-sphere. The ranges of the angular coordinates are explicitly $0 \leq \tilde{\theta} \leq \pi$ and $0 \leq \tilde{\phi} < 2\pi$, while $0 \leq \theta \leq \pi/2$ and $0 \leq \phi^1, \phi^2 < 2\pi$. The dilaton is

$$e^{-2\Phi+2\Phi_0} = e^{2g^2 z} \left(1 - \sin^2 \theta \frac{1 + k e^{-2g^2 z}}{2g^2 z} \right), \tag{17}$$

and the NS three-form reads

$$\begin{aligned}
 H = & \frac{2 \sin \theta \cos \theta}{g^2 \Omega^2} \left(\sin \theta \cos \theta \frac{dx}{dz} dz - d\theta \right) \wedge (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi}) \wedge d\phi_2 \\
 & + \frac{e^{-x} \sin^2 \theta}{g^2 \Omega} \sin \tilde{\theta} d\tilde{\theta} \wedge d\tilde{\phi} \wedge d\phi_2.
 \end{aligned} \tag{18}$$

The solution depends on two parameters: the expectation value of the dilaton Φ_0 and the parameter k appearing in the function $x(z)$ that is given in (14) and plotted in Fig. 1.

3.3 Symmetries of the solution

As solutions of type IIB supergravity, we expect that the solution preserves eight supercharges. To see this we determine the number of Killing spinors by setting the supersymmetry variations of the fermions to zero. For vanishing RR fields we require

$$\begin{aligned}\delta\lambda &= \Gamma^\mu\partial_\mu\Phi\tau_3\boldsymbol{\eta} - \frac{1}{12}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\boldsymbol{\eta} = 0, \\ \delta\psi_\mu &= \nabla_\mu\boldsymbol{\eta} - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\tau_3\boldsymbol{\eta} = 0,\end{aligned}\tag{19}$$

where $\boldsymbol{\eta}$ is the $SO(2)$ -doublet of chiral IIB supersymmetry parameters and τ_3 is the third Pauli matrix. Using a slightly non-obvious frame given in the appendix, and inspired by that in [11], we find that the Killing spinors are given by

$$\boldsymbol{\eta} = e^{-\frac{1}{2}(\phi_1\Gamma^{67}+\phi_2\Gamma^{89})}\boldsymbol{\eta}_0\tag{20}$$

where $\boldsymbol{\eta}_0$ is a constant spinor satisfying

$$\begin{aligned}\Gamma^{6789}\boldsymbol{\eta}_0 &= -\tau_3\boldsymbol{\eta}_0 \\ \Gamma^{4567}\boldsymbol{\eta}_0 &= -\boldsymbol{\eta}_0.\end{aligned}\tag{21}$$

This is exactly as desired with eight independent Killing spinors satisfying the conditions (21). Furthermore, the first projection is the same as that for a NS fivebrane with tangent directions $\{0, 1, 2, 3, 4, 5\}$, while the second is the same as that for a Killing spinor on a Calabi–Yau two-fold with tangent directions $\{4, 5, 6, 7\}$. In other words, the supersymmetry preserved matches that of a NS fivebrane probe wrapping a two-cycle in a Calabi–Yau two-fold as expected. Note that as a solution of type I or type IIA string theory, the solution similarly preserves 1/4 of the supersymmetry.

The non-trivial part of the solution is six-dimensional and thus is a non-compact example of the class of solutions with torsion discussed in [34]. In particular the six-dimensional space should be a complex manifold, with the complex structure J being constructed from the Killing spinors. For our solution, the corresponding two-form K constructed by lowering one index of J can be written as

$$\begin{aligned}K &\equiv \frac{1}{2}J_{ab}e^a\wedge e^b \\ &= e^4\wedge e^5 + e^6\wedge e^7 \pm e^8\wedge e^9,\end{aligned}\tag{22}$$

where we have directly checked that such a J is indeed integrable. It will be useful later to note that six-form dual potential \tilde{B} , defined by

$$d\tilde{B} = e^{-2\Phi} * H, \quad (23)$$

is given by

$$\tilde{B} = \text{Vol}_4 \wedge e^{-2\Phi} K, \quad (24)$$

where Vol_4 is the volume form on the flat world volume directions.

We have seen that the solutions thus preserve the right amount of supersymmetry to correspond to the supergravity duals of pure $\mathcal{N} = 2$ super Yang–Mills theory. Let us now discuss how the R -symmetry of the gauge theory arises as a symmetry of the solution. Recall that the classical gauge theory has $SU(2) \times U(1)$ R -symmetry that is broken down to $SU(2) \times Z_{4N}$ by anomalies in the quantum theory. The construction we have used, incorporating the appropriate twistings, automatically has $U(1) \times U(1) = SO(2)_1 \times SO(2)_2$ isometries. These are just shifts in ϕ_1 and ϕ_2 . Recall from section two that all the fields in the $D = 4$, $\mathcal{N} = 2$ gauge multiplet are singlets under $SO(2)_1$ since the two-sphere is rigid within the Calabi–Yau manifold. Thus this first factor should be irrelevant in the IR. However, the scalar fields transform as a doublet under $SO(2)_2$, so we conclude the second factor corresponds to the $U(1)$ R -symmetry of the classical gauge theory – we shall remark later on its breaking down to a Z_{4N} subgroup. Actually since our solution includes a round two-sphere, corresponding to the cycle on which the fivebrane is wrapped, the isometry group is larger. If we introduce a set of $SO(3)$ left-invariant one-forms in terms of the Euler angles $\{\tilde{\theta}, \tilde{\phi}, \phi_1\}$

$$\begin{aligned} \sigma_1 &= \cos \phi_1 d\tilde{\theta} + \sin \phi_1 \sin \tilde{\theta} d\tilde{\phi} \\ \sigma_2 &= -\sin \phi_1 d\tilde{\theta} + \cos \phi_1 \sin \tilde{\theta} d\tilde{\phi} \\ \sigma_3 &= d\phi_1 + \cos \tilde{\theta} d\tilde{\phi}, \end{aligned} \quad (25)$$

we see that full isometry group is $SO(3) \times U(1)^2$. (Notice that since $0 \leq \phi_1 < 2\pi$, the manifold parametrised by σ_i is topologically S^3/Z_2 and hence we have $SO(3)$ symmetry and not $SU(2)$.) This “accidental” $SO(3)$ isometry would not arise if we were to wrap around other two-surfaces. Instead, in general, all that would survive is a $SO(2)$ symmetry corresponding to rotations of the tangent space to the cycle. Since this $SO(3)$ cannot provide the remaining $SU(2)$ R -symmetry of the $\mathcal{N} = 2$ gauge theory, we see that this symmetry cannot arise purely from isometries of the solution.

Recall that in the gauge theory the two supercharges form a doublet of the $SU(2)$ R -symmetry. As a consequence we expect to see this symmetry acting on the Killing spinors of our solution. These can include transformations which rotate between the two ten-dimensional chiral IIB spinors. We find that the transformations consistent with (21) are generated by Γ_{89} , corresponding to $U(1)_R$, and also $\Gamma^{45}, \Gamma^{48}\tau_1, \Gamma^{58}\tau_1$, where τ_1 is the first Pauli matrix, which do indeed generate $SU(2)_R$.

3.4 UV and IR behaviour of solutions

Let us continue our analysis of the solutions by examining the asymptotic behaviour of the solutions. The UV limit is obtained when $z \rightarrow \infty$ and hence $x \rightarrow 0$. The metric and the dilaton are given by

$$\begin{aligned} ds^2 = d\xi_{1,3}^2 + z \left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right) + g^2 dz^2 + \frac{1}{g^2} d\theta^2 \\ + \frac{1}{g^2} \sin^2 \theta d\phi_2^2 + \frac{1}{g^2} \cos^2 \theta \left(d\phi_1 + \cos \tilde{\theta} d\tilde{\phi} \right)^2, \quad (26) \\ e^{-2\Phi+2\Phi_0} = e^{2g^2 z}. \end{aligned}$$

This has the same form as the near horizon limit of the flat NS-fivebrane solution (1) but with world volume $\mathbb{R}^{1,3} \times S^2$ instead of $\mathbb{R}^{1,5}$ and the appropriate twisting. Notice that the size of the S^2 is diverging, as in [2], which is related to the fact that $\mathcal{N} = 2$ Yang–Mills theory is asymptotically free, since on dimensional reduction from six dimensions the four-dimensional coupling is inversely proportional to the volume. In addition we can connect the seven dimensional gauge coupling g^2 to the number N of wrapped branes via:

$$\frac{1}{g^2} = N. \quad (27)$$

This relation can be seen directly by noting that the integral of the three-form $H/4\pi^2$ over the transverse three-sphere, which gives the number N of NS-fivebranes, is equal to g^{-2} . This will be useful later, when comparing with dual gauge theory expectations.

The one parameter family of solutions, specified by k , are all singular in the IR. For $k \geq -1$ the range of the radial variable z is $z_0 \leq z \leq \infty$ where z_0 is the solution of $e^{-2x(z_0)} = 0$ (see Fig. 1). When $k = -1$ we have $z_0 = 0$. For these values of k the solutions are singular when $z = z_0$ and $\theta = \pi/2$: this can be seen from the behaviour of the dilaton, for example. On the other hand when $k < -1$ we have $0 \leq z \leq \infty$ and the solutions are singular when $z \rightarrow 0$ for generic θ . We have plotted the behaviour of the dilaton for $\theta = \pi/2$ in Fig. 2.

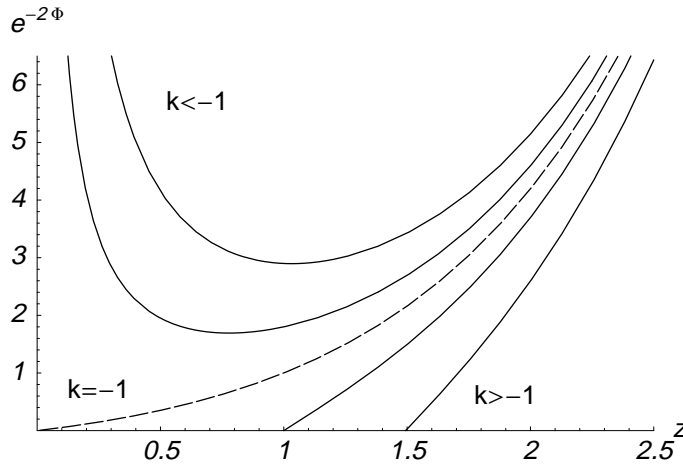


Figure 2: Behaviour of the dilaton at $\theta = \pi/2$.

We would like to argue that the $k \geq -1$ solutions are related to gravity dual descriptions of part of the Coulomb branch of $\mathcal{N} = 2$ Yang–Mills theory, while the $k < -1$ solutions appear to be unphysical. The first piece of evidence for this is that the singularities are “good” for $k \geq -1$ and “bad” for $k < -1$, using the criteria of [2]. Recall that the criteria is based on the behaviour of the norm of the time-like Killing vector field in the Einstein frame. If the norm is decreasing as one approaches the singularity, as it is for $k \geq -1$, fixed proper-energy excitations in the geometry correspond to smaller and smaller energy excitations in a possible dual field theory, which is consistent with the singularity corresponding to the far IR physics. For $k < -1$ this norm increases and the singularity would need excising in some way if one was to develop a similar interpretation⁴.

A second piece of evidence that $k \geq -1$ corresponds to the Coulomb branch comes from analysing the metric near the singularity at $z = z_0$ and $\theta = \pi/2$. We have $e^{-2x} \approx 2g^2(z - z_0)$, and defining new variables y and ψ by

$$\begin{aligned}\sqrt{2g^2(z - z_0)} &\equiv \sqrt{2gy} \sin(\psi/2) \\ \theta - \pi/2 &\equiv \sqrt{2gy} \cos(\psi/2)\end{aligned}\tag{28}$$

⁴Note there are solutions with “bad” singularities that ultimately get resolved ,e.g., [2]. However, unlike that example there is no reason to expect this to occur here.

with $0 \leq \psi < \pi$ and $y \geq 0$ so that $z \geq z_0$ and $\theta \leq \pi/2$, we find

$$\begin{aligned} ds_6^2 &= z_0 \left(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right) \\ &\quad + (2gy)^{-1} \left[dy^2 + y^2 \left(d\psi^2 + \sin^2 \psi (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi})^2 \right) + g^{-2} d\phi_2^2 \right] \quad (29) \\ e^{-2\Phi+2\Phi_0} &= 2gy e^{2g^2 z_0} \end{aligned}$$

with the singularity located at $y \rightarrow 0$. We see that the metric has precisely the form of the near-horizon limit of $g^{-2} = N$ NS 5-branes smeared on a circle in the ϕ_2 direction. The fact that the coordinate direction ϕ_2 is singled out is supported by our construction. Recall that the twisting we implemented corresponds to a NS fivebrane wrapping a two-cycle in a Calabi–Yau two-fold. Roughly, a radial direction built from z and θ and the angle ϕ_1 correspond to the two directions transverse to the fivebrane and inside the Calabi–Yau two-fold. A different radial coordinate together with the angle ϕ_2 correspond to the two directions transverse to the fivebrane and also to the Calabi–Yau two-fold. (In fact, this can be made exact by an explicit change of variables similar to that given in [11].) From the discussion in section two, it is the latter directions which are related to the Coulomb branch of vacua. It appears that the solutions correspond to a slice of the Coulomb branch where the branes are smeared in a ring parametrised by ϕ_2 . For $k = -1$ it is less clear how to make this direct argument: the volume of the two sphere that the fivebrane wraps is shrinking to zero size at the singularity and it is thus difficult to compare to the smeared fivebrane solution. Our interpretation is that the $k = -1$ solution corresponds to the smallest radius, namely zero, on which the fivebranes can be distributed. Further confirmation of this picture will be developed in the next section when we study the dynamics of a fivebrane probe.

3.5 Anomaly in $U(1)_R$

We have noted that the supergravity solution has a $U(1)$ isometry, corresponding to shifts in ϕ_2 , that can be identified with the $U(1)$ R -symmetry of classical $\mathcal{N} = 2$ Yang–Mills theory. In the quantum gauge theory only a Z_{4N} subgroup is anomaly free, with a Z_{2N} acting on the moduli space of vacua (see e.g. [35]). As in [3] we expect that the anomaly can only be seen by going beyond the supergravity approximation and by incorporating string world-sheet instantons. In particular consider a string worldsheet wrapping the two-sphere parametrised by $\tilde{\theta}, \tilde{\phi}$ at $\theta = \pi/2$. The flux of the B -field over the two sphere is a function of ϕ_2 , and at $\theta = \pi/2$, we have, by repeating

the argument in [3],

$$\frac{1}{2\pi} \int_{\phi_2} B = b - 2N\phi_2 \quad (30)$$

for some constant b . This flux appears in the worldsheet instanton calculation and has period 2π . Identifying this with the field theory θ_{FT} angle (which has the same period), we find the anomaly free Z_{4N} subgroup of the $U(1)_R$ -symmetry: shifts in ϕ_2 by $\pi n/N$ with $1 \leq n \leq 4N$ will not change θ_{FT} . Note that this is Z_{4N} since although ϕ_2 has period 2π , the fermions pick up a minus sign (as can be seen, for example, from (20)), and are only periodic under $\phi_2 \rightarrow \phi_2 + 4\pi$.

3.6 D5 solution

To decouple the four-dimensional gauge theory it is necessary to go to scales much smaller than the little string theory mass scale. In this limit the dilaton becomes large and so we should consider the S-dual solution corresponding to wrapped D-fivebranes [28]. The D5 brane solution is obtained via the following transformation rules

$$\begin{aligned} \Phi_{D5} &= -\Phi_{NS5}, \\ ds^2(D5) &= e^{-\Phi_{NS5}} ds^2(NS5), \\ C_{(2)} &= -B, \end{aligned} \quad (31)$$

where the quantities on the left hand side refer to the S-dual dilaton, metric, and RR potentials. Similarly the corresponding six-form potential dual to $C_{(2)}$ is given by $C_{(6)} = -\tilde{B}$, which we will use in the next section.

4 Probing the solution with fivebranes

A standard technique for analysing the physics of supergravity solutions is to study the low-energy dynamics of a probe brane. For definiteness, we will probe the wrapped D5-brane solutions given at the end of the last section. For these solutions it is natural to consider a D5-brane probe wrapping the same two-sphere. Physically, one expects that this corresponds to pulling out one of the N D5 microscopic constituents and studying its low-energy dynamics. From the dual gauge theory point of view this process corresponds to Higgsing $SU(N) \rightarrow SU(N-1) \times U(1)$, and therefore we expect to find an effective action corresponding to the $U(1)$ factor, due to the background of the remaining branes. As we will see, we do indeed find a two-dimensional moduli space with dynamics governed by a holomorphic prepotential that incorporates the

perturbative effects of the gauge theory. Moreover for $k > -1$ we find a non-singular moduli space despite the geometry being singular. For $k = -1$ we will see that the kinetic energy of the probe brane is tending to zero in the IR.

The effective action of a probe D5 brane in string frame reads, using the conventions of [24],

$$S = -\mu_5 \int d^6 y e^{-\Phi_{D5}} \sqrt{-\det [G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}]} + \mu_5 \int [\exp(2\pi\alpha' F + B) \wedge \oplus_n C_{(n)}] , \quad (32)$$

where $\mu_5 = (2\pi)^{-5} \alpha'^{-3}$, F_{ab} is a world volume Abelian gauge field, B is the NS two-form, $C_{(n)}$ are the RR n -forms and it is understood that ten-dimensional fields are pulled back to the six-dimensional world volume. For the D5-brane solution (31) we have $B = 0$, but non-vanishing $C_{(2)}$ and $C_{(6)}$.

We choose the world-volume to have topology $\mathbb{R}^{1,3} \times S^2$ and fix the reparametrisation invariance by identifying the world-volume coordinates with $\{\xi^i, \tilde{\theta}, \tilde{\phi}\}$ in space-time. The four scalar fields z , θ , ϕ_1 and ϕ_2 are then functions of these coordinates. The dynamics of the fivebrane that are relevant for our purposes are when the fields are just dependent on the four-dimensional coordinates $\{\xi^i\}$. To get the full effective theory we will also consider non-zero four-dimensional gauge fields. First, though, let us suppose the probe brane is at rest so that z , θ , ϕ_1 and ϕ_2 are constants and set $F = 0$. With this restriction we find that, in general there is a non-zero potential arising from the DBI and the six-form contribution to (32). The contribution to the effective action is explicitly

$$S_{\text{potential}} = -\mu_5 e^{-2\Phi_0} \int d^4 \xi d\tilde{\theta} d\tilde{\phi} \Omega e^{-x} z e^{2g^2 z} \sin \tilde{\theta} \left[1 - \left(1 + \cos^2 \theta \Omega g^2 z e^x \tan^2 \tilde{\theta} \right)^{1/2} \right] . \quad (33)$$

For there to be no net force on the probe brane we must be at a minimum of this potential. We find two loci

$$\begin{aligned} \text{I : } & \theta = \pi/2 \quad \text{for all } k, \\ \text{II : } & z = z_0 \quad \text{for } k \geq -1, \end{aligned} \quad (34)$$

for which the potential, in fact, vanishes. It is straightforward to check that both of these configurations preserve 1/4 supersymmetry.

The dimensionality of these moduli spaces can be determined by considering the kinetic energy terms of the scalar fields arising from the DBI part of (32), where

we now let z , θ , ϕ_1 and ϕ_2 be functions of ξ^i . Writing $\partial_i = \partial/\partial\xi^i$, we have, after integrating over the two-sphere, on locus I

$$S_{\text{scalar}} = -2\pi\mu_5 e^{-2\Phi_0} \int d^4\xi z e^{2g^2 z} \left[g^2 (\partial z)^2 + \frac{1}{g^2} (\partial\phi_2)^2 \right] \quad (\text{I}), \quad (35)$$

independent of k , and on locus II

$$S_{\text{scalar}} = -2\pi\mu_5 e^{-2\Phi_0} g^2 \int d^4\xi z_0 e^{2g^2 z_0} [\cos^2 \theta (\partial\theta)^2 + \sin^2 \theta (\partial\phi_2)^2] \quad (\text{II}). \quad (36)$$

In each case we have a two-dimensional moduli space as we expect for the Coulomb branch of Abelian $\mathcal{N} = 2$ super Yang–Mills theory. The moduli space metric on locus I using the variable $g^2 z = \log w$ becomes

$$ds_{\mathcal{M}_I}^2 = \frac{4\pi\mu_5 e^{-2\Phi_0}}{g^4} \log w (dw^2 + w^2 d\phi_2^2). \quad (37)$$

while on locus II using $\zeta = \sin \theta$, and recalling that $0 \leq \theta \leq \pi/2$, we obtain the flat disc

$$ds_{\mathcal{M}_{II}}^2 = R^2 (d\zeta^2 + \zeta^2 d\phi_2^2). \quad (38)$$

of radius $R^2 = 4\pi\mu_5 g^{-2} e^{-2\Phi_0} z_0 e^{2g^2 z_0}$.

When $k > -1$ the two loci intersect along the ring $z = z_0, \theta = \pi/2$, which have exactly the same radius on each loci. Thus we see that the union of the two loci have the topology of a plane, and the tension of the probe is finite on the ring. In other words, quite remarkably, we obtain a non-singular two dimensional moduli space despite the singularity located at the ring in the supergravity solution.

Recall that the $k = -1$ solution is a limiting solution of the $k > -1$ solutions in the sense that the singularity is still good. From the probe point of view one finds that the locus II degenerates and the kinetic energy terms of the probe brane become zero when $z = 0$ or equivalently $w = 1$. Note that similar behaviour is also realised by the probe for the apparently non-physical $k < -1$ solutions.

To complete the four-dimensional effective probe action, now consider the gauge fields on the brane. Since we are wrapping a sphere, the only gauge field zero modes come from F with components along the $\mathbb{R}^{1,3}$ directions of the wrapped probe.]For the WZ part of the probe action we need $C_{(2)}$ which is given by $-B$ of the NS solutions. Starting with (18) note that we can write

$$B = -d \left[\sin^2 \theta g^2 e^x \Omega (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi}) \right] \phi_2 \quad (39)$$

which respects the periodicity of ϕ_2 after performing a gauge transformation on B . Keeping terms from both the DBI and the WZ part of the probe action, after integrating over S^2 , we find on locus I

$$S_{\text{gauge}} = -\frac{\pi(2\pi\alpha')^2\mu_5}{g^2} \int d^4\xi \left(\log w F^2 + \phi_2 F \tilde{F} \right) \quad (\text{I}). \quad (40)$$

Note that for $k = -1$, as we approach $z = 0$ the kinetic energy terms of the gauge fields are dropping to zero. For locus II we find

$$S_{\text{gauge}} = -\pi(2\pi\alpha')^2\mu_5 \int d^4\xi \left(z_0 F^2 + c_0 F \tilde{F} \right) \quad (\text{II}), \quad (41)$$

where \tilde{F} is the Hodge dual of F and c_0 is a constant. Note that these results are independent of the dilaton, which cancels against the contribution from the determinant.

The $\mathcal{N} = 2$ supersymmetry implies that the full action should have the form

$$S = \frac{1}{8\pi} \int d^4\xi \left(-\text{Im} \tau(u) (\partial u) (\partial \bar{u}) + \frac{1}{2} \text{Re} \left[\tau(u) \left(iF^2 + F \tilde{F} \right) \right] \right) \quad (42)$$

where u is the complex scalar field in the $\mathcal{N} = 2$ vector multiplet and the Yang–Mills coupling $\tau(u) \equiv (\Theta_{YM}/2\pi) + i(4\pi/g_{YM}^2)$ is a holomorphic function of u . Comparing with the expressions for the scalar and gauge field actions (35), (36), (40) and (41), using $g^{-2} = N$, and setting $\alpha' = 1$ we can identify on locus I

$$\tau = i \frac{2N}{\pi} \log(u/\Lambda) \quad (\text{I}), \quad (43)$$

where $u = \Lambda w e^{i\phi_2}$ with $\Lambda = \sqrt{N}/2\pi e^{\Phi_0}$. On locus II, the complex scalar is given by $u = \Lambda e^{g^2 z_0} \zeta e^{i\phi_2}$ we find τ is constant,

$$\tau = \tau_0 \equiv i \frac{2N}{\pi} \log(u_0 e^{ic_0}/\Lambda) \quad (\text{II}), \quad (44)$$

where u_0 is the value of $|u|$ at $z = z_0$. Note that the logarithmic behaviour of τ on locus I is very reminiscent of the exact perturbative behaviour in $\mathcal{N} = 2$ super Yang–Mills theory. In the next section we will make a more precise comparison.

5 Comparison with gauge theory

To complete the comparison of the probe effective action calculated above with that expected from the dual field theory, let us now derive the expected form of $\tau(u)$ from gauge theory. This will be a perturbative calculation and will depend on where

exactly we are on the Coulomb branch moduli space. In particular, we will show that our result is compatible with being at a slice where the branes are distributed on a ring at $|u| = u_0$.

Following the discussion in [24], for $SU(N)$ gauge theory the Coulomb branch moduli space is parametrised by the $N - 1$ independent complex expectation values of the adjoint scalar, representing the relative positions of the N branes,

$$\Phi = \text{diag}(a_1, \dots, a_N), \quad (45)$$

where for $SU(N)$ we have $\sum_i a_i = 0$. For generic $\{a_i\}$ the theory is broken to $U(1)^{N-1}$ and the bosonic low-energy effective action is given by the generalisation of (42), namely

$$S = \frac{1}{8\pi} \int d^4\xi \left(-\text{Im} \tau_{ij} \partial a^i \partial \bar{a}^j + \frac{1}{2} \text{Re} \left[\tau_{ij} \left(i F^i F^j + F^i \tilde{F}^j \right) \right] \right). \quad (46)$$

The couplings τ_{ij} are given in terms of a holomorphic prepotential \mathcal{F} ,

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}. \quad (47)$$

Perturbatively the prepotential is given by

$$\mathcal{F} = \frac{i}{8\pi} \sum_{i \neq j} (a_i - a_j)^2 \log \frac{(a_i - a_j)^2}{\mu^2}, \quad (48)$$

and is one-loop exact. There are in addition, non-perturbative instanton corrections. However, provided $|a_i - a_j| > O(1/N)$ it was argued in [24] that these corrections vanish in the large N limit.

Now consider the calculation in the last section of the dynamics of the probe brane in our supergravity background. In field theory this should correspond to breaking a $SU(N + 1)$ theory to $U(1)^{N-1} \times U(1)$ with the first factor corresponding to the background and the second to the probe brane. If we write u for the position of the probe brane we have as in [24]

$$\Phi = \text{diag}(u, a_1 - u/N, a_2 - u/N, \dots, a_N - u/N). \quad (49)$$

In the large N limit we then get

$$\tau(u) = \frac{\partial^2 \mathcal{F}}{\partial u^2} = \frac{i}{2\pi} \sum_i \log \frac{(u - a_i)^2}{\mu^2}. \quad (50)$$

Since N is large we can replace the sum by an integral

$$\tau(u) = \frac{i}{2\pi} \int d^2a \rho(a) \log \frac{(u-a)^2}{\mu^2}, \quad (51)$$

with the density function $\rho(a)$ normalized by $\int d^2a \rho(a) = N$.

From our analysis of the supergravity solution we expect the field theory dual should have the branes arrayed on a ring at radius $|u| = u_0$ so $\rho(a) = (N/2\pi u_0) \delta(|a| - u_0)$. Integrating, this gives

$$\tau(u) = i \frac{N}{\pi} \log(u/\mu) \quad \text{for } |u| \geq u_0, \quad (52)$$

and

$$\tau(u) = i \frac{N}{\pi} \log(u_0/\mu) \quad \text{for } |u| \leq u_0. \quad (53)$$

We see that, up to an overall normalization factor of two, this matches precisely the form calculated for a probe brane with $k \geq -1$ in both locus I and locus II. This normalisation presumably corresponds to a normalisation of the gauge fields.

6 Discussion

We have presented exact supergravity duals describing fivebranes wrapped on a two-sphere that correspond in the IR to a slice of the Coulomb branch of $\mathcal{N} = 2$ super Yang–Mills theory. We have shown that the solutions have the appropriate symmetries including the fact that the $U(1)$ R -symmetry is broken to a discrete group by string world-sheet instantons. We have also shown that the IR singularities correspond to the wrapped fivebranes being uniformly distributed on a ring and that the dynamics of a probe fivebrane incorporates the full perturbative effects expected from the gauge theory.

It is interesting to compare our results with those of [17], where a one parameter family of supergravity solution was presented corresponding to a slice of the Coulomb branch of the $\mathcal{N} = 2^*$ theory. This theory arises from mass deformations of $\mathcal{N} = 4$ super Yang–Mills theory and thus the supergravity solution can be considered to be made from deformed three-branes. There are a number of similarities with our solutions. The solutions parametrised by γ in [17] should be compared with ours as follows: $\gamma \leq 0$ with $k \geq -1$ and $\gamma > 0$ with $k < -1$. The former solutions appear to be physical while those with $\gamma > 0$ do not (at least as far as being dual to $\mathcal{N} = 2^*$ Yang–Mills theory). The dynamics of a D3-brane probe studied in [24, 25] found

for $\gamma < 0$ that the moduli space similarly had two loci. One of the loci, labelled locus II, is a flat disk which degenerated to a line segment for $\gamma = 0$. The metric in the supergravity solution is singular on this locus, corresponding to a distribution of D3-branes over the disk. This is in contrast to our case where the fivebranes are distributed on a ring for $k \geq -1$. As in our analysis for $k < -1$, the moduli space for the probe-brane is completely regular for $\gamma < 0$ despite the presence of singularities in the solution. Another similarity is that the dynamics for $\gamma = 0$ has the feature that kinetic energy terms of the D3-brane probe are tending to zero as one approached the singularity. We have observed exactly the same features here for $k = -1$.

It would be interesting if one could find generalised supergravity solutions corresponding to more general slices of the Coulomb branch in the gauge theory. In our approach this will require relaxing the ansatz that we considered. There are several obvious directions, for example incorporating more scalar fields in the $D = 7$ gravity ansatz, which could then include non-circularly symmetric configurations, but it is not clear that exact solutions could be found.

By considering the dynamics of a probe fivebrane we have argued that our supergravity solutions include the full perturbative effects of the gauge theory. To probe the structure of Seiberg–Witten theory [36] one would also need to include instanton effects. As noted in [24] in the large N limit one needs to be considering the physics where the vevs $|a_i - a_j|$ are smaller than order $1/N$ and it would be interesting if such effects could be incorporated in a supergravity dual.

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A Orthonormal Frame

The supersymmetry preserved by the fivebrane is best demonstrated in a slightly non-obvious orthonormal frame. We choose

$$\begin{aligned}
e^i &= dx^i & i = 0, \dots, 3 \\
e^4 &= z^{1/2} d\tilde{\theta} \\
e^5 &= z^{1/2} \sin \tilde{\theta} d\tilde{\phi} \\
e^6 &= \frac{1}{\Omega^{1/2}} (-g e^{3x/2} \cos \theta dz + \frac{e^{-x/2}}{g} \sin \theta d\theta) \\
e^7 &= \frac{e^{-x/2}}{g \Omega^{1/2}} \cos \theta (d\phi_1 + \cos \tilde{\theta} d\tilde{\phi}) \\
e^8 &= -\frac{e^{x/2}}{\Omega^{1/2}} (g \sin \theta dz + \frac{1}{g} \cos \theta d\theta) \\
e^9 &= \frac{e^{x/2}}{g \Omega^{1/2}} \sin \theta d\phi_2
\end{aligned} \tag{54}$$

where Ω and $x(z)$ are given in the main text. This differs from the obvious orthonormal by including a rotation between the z and θ tangent directions. A similar kind of frame was found to be useful in a related context in [11].

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